# On the Synthesis and Realization of Selective Linear Phase IIR Filters

Dejan Mirković, Ivan Litovski and Vančo Litovski

***Abstract*—This paper gives an overview of the procedure of design of selective linear-phase IIR digital filters. Synthesis of minimum-phase transfer functions will be described. Two approximation criteria will be implemented for constant group delay approximation: the maximally flat and the equi-ripple. The synthesis in the s-domain will be performed in two steps. First, a polynomial function exhibiting linear phase will be synthesized and then transmission zeros located on the imaginary will be added. The IIR filter will be obtained by bilinear transform. To avoid stability problems and, in the same time, problems related to the signal dynamics within the filter, parallel implementation of the IIR filter will be used. Two examples will be given exemplifying the implementation of the two approximation criteria used.**

***Index Terms*—IIR filters, linear phase, selective digital filters.**

## I. Introduction

THE linear phase property of digital filters is usually attributed to the FIR filters [1]. That, however, is achieved by doubling the number of coefficients in the filter function what has, as a consequence, doubling of the hardware needed for realization. In such a solution the phase linearity is extended over the stop-band which is not necessary and reveals the notion of unnecessary redundancy. In addition, there are severe restrictions to the shape of the amplitude characteristic when using FIR filters. Here we offer a procedure for selective linear phase IIR filter design where both, the amplitude and phase characteristics, are under full control of the designer. A design procedure, that starts in the s-domain and after bilinear transformation produces the filter coefficients, will be described. The procedure is alternative to the one described (For maximally-flat delay, only) in [2, 3] where the whole process is performed in the z-domain. Parallel implementation of the IIR filter will be used. The examples demonstrating the design process will exhibit maximally-flat and equi-ripple approximation of the group delay while the selectivity of the filter will be achieved by inserting imaginary-axis zeros which, in parallel realization, does not affect the hardware complexity. The final results will be expressed as sixteen-bit normalized two's complements.

The paper is organized as follows. First, in the second paragraph, we describe the constant group delay of low-pass filters in the s-domain. Then, we introduce the algorithm of transmission zeros inserting at the imaginary axis. The s-domain is then abandoned and in the fourth paragraph we describe the use of the bilinear transform for synthesis of IIR filters with parallel implementation. Finally, in the fifth paragraph we will report two design examples illustrating the method and giving insight into the properties of the functions obtained.

## II. Constant Group Delay Approximation

Lack of liner phase (i.e. constant group delay) is the most commonly stated fact in favor to FIR filters when compared to IIR one. Accordingly, group delay compensation is inevitable with IIR filters whenever linear phase is sought [4, 5]. However, the problem of constant group delay approximation of low-pass analog filters is already solved. In [6, 7, 8] one may find procedures to achieve both. That is why we will not go for the algorithms here. Instead we will demonstrate the results.

To synthesize a low-pass filter exhibiting equi-ripple approximation of the group delay in the s-domain, we wrote a program in C. Results obtained by implementation of that program are illustrated in Fig. 1 for an eight order polynomial low-pass filter. In this special case an approximation error of ±1% was allowed. Fig. 2 depicts the group delay of this filter only.

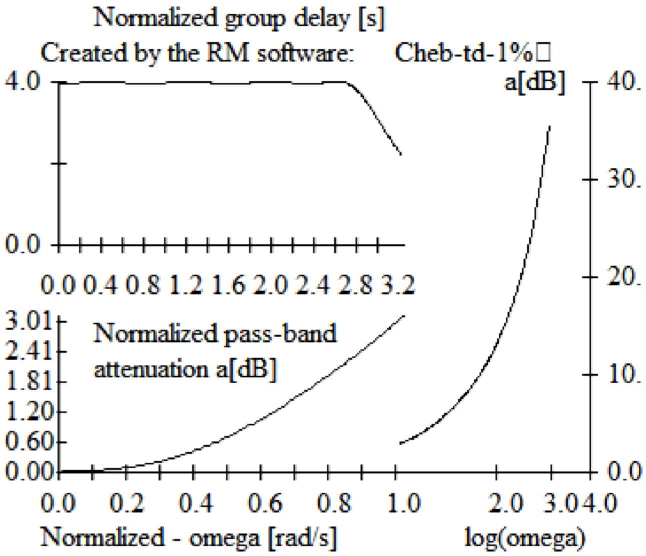


Fig. 1. Attenuation and group delay response of an eight order polynomial filter approximating constant group delay in equi-ripple manner

Table I summarizes data about selectivity of all filters presented in this paper. Since all functions are normalized so

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that at ω=1 rad/s to have an attenuation of 3 dB, the main information about the selectivity is the width of the transition region defined for some stop-band attenuation level. Here we have chosen the value of 40 dB as the minimum allowable stop-band attenuation.

TABLE I. TRANSITION REGION WIDTH (NORMALIZED, IN RAD/S)

| Type of function→ / Type of approximation↓ | Polynomial | Rational | Reduction (%) |
|---|---|---|---|
| Maximally flat | 2.342017 | 1.817169 | 28.894 |
| Equi-ripple | 2.126709 | 1.797034 | 18.345 |

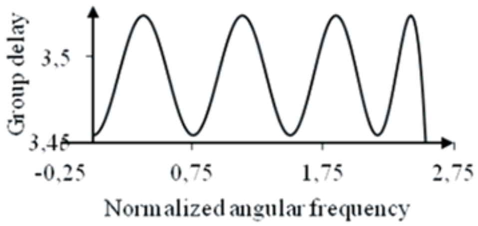


Fig. 2. Group delay response of an eight order polynomial filter approximating constant group delay in equi-ripple manner

As an alternative a maximally flat approximation of the group delay was performed using another program. Bessel polynomials were used as first introduced in [9]. The properties of the resulting polynomial filter in the s-domain are graphically depicted in Fig. 3, while Table I expresses the selectivity property of the solution.

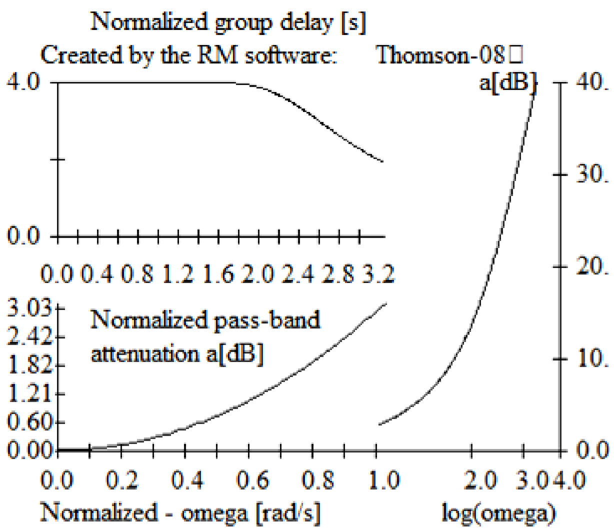


Fig. 3. Attenuation and group delay response of an eight order polynomial filter approximating constant group delay in maximally flat manner

As can be seen from Table I, the equi-ripple solution exhibits slightly higher selectivity. That comes from the fact that less restrictive requirements were imposed for the group delay approximation. Namely, if zero valued error was sought, the equi-ripple solution would be reduced to the maximally flat one.

## III. TRANSMISSION ZEROS INSERTION

The algorithm for insertion of finite transmission zeros at the imaginary axis of the complex frequency plane is already known [10]. It will be not repeated here. To make the proceedings clear, however, some definitions will be given.

The transfer function of an all-pole analog filter may be expressed as follows

$$H(s) = \frac{\prod_{i=1}^{m}(-p_i)}{\prod_{i=1}^{m}(s-p_i)}, \tag{1}$$

where $p_i = \sigma_i + j\cdot\omega_i$, $i$=1,2,...,$m$, are the complex poles of the transfer function, $m$ is its order, and $s$ is the complex frequency. This function has all zeros at infinity.

When zeros at the imaginary axis are introduced the group delay characteristic of the filter remains unchanged apart for the renormalization necessary to be done after the approximation process is finished. The new transfer function will have the form

$$H(s) = \frac{\prod_{i=1}^{m}(-p_i)}{\prod_{i=1}^{m}(s-p_i)} \times \frac{\prod_{k=1}^{n/2}(s^2+\omega_k^2)}{\prod_{k=1}^{n/2}(\omega_k^2)} \tag{2}$$

where $\omega_k$, $k$=1,2,..,$n$/2 is the abscissa of the zero at the imaginary axis, and $n$ is an even number. $n<m$.

Note, in both formulae the gain at the origin is reduced to unity.

After insertion of transmission zeros and proper renormalization of the amplitude characteristic is performed the following results were obtained.

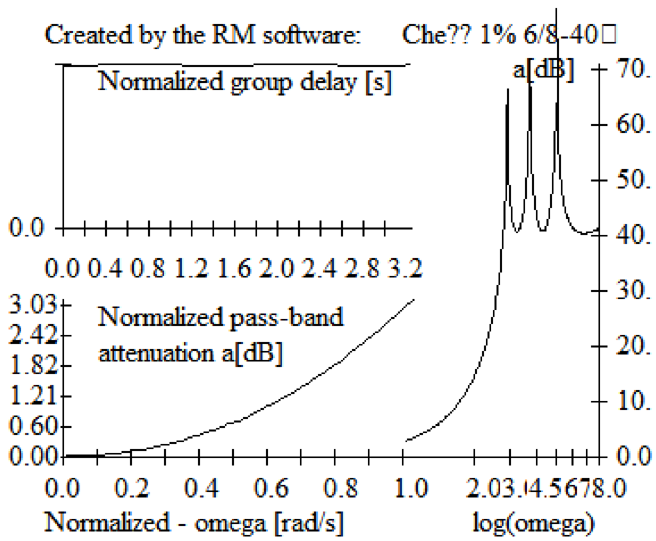


Fig 4. Attenuation and group delay response of the rational transfer function approximating constant group delay in equi-ripple manner.

Fig. 4. represents the attenuation and group delay response of the new transfer function approximating constant group delay in equi-ripple manner. Minimum stop-band attenuation of 40 dB was required. $n$=6 was used. The final positions of the zeros and poles of this filter are given in Table II.

TABLE II. RATIONAL FUNCTION APPROXIMATING CONSTANT GROUP DELAY IN EQUI-RIPPLE MANNER. N=6, M=8

| | Real part | Imaginary part |
|---|---|---|
| Zeros | 0.000000 | ±2.917597 |
| | 0.000000 | ±3.738559 |
| | 0.000000 | ±5.046967 |
| Poles | -1.177311 | ±0.681299 |
| | -1.152160 | ±2.029670 |
| | -1.077362 | ±3.327213 |
| | -0.837877 | ±4.516940 |

Similarly, Fig. 5. represents the attenuation and group delay response of the new transfer function approximating constant group delay in maximally flat manner. Minimum stop-band attenuation of 40 dB was required. $n$=6 was used. The final positions of the zeros and poles of this filter are given in Table III.

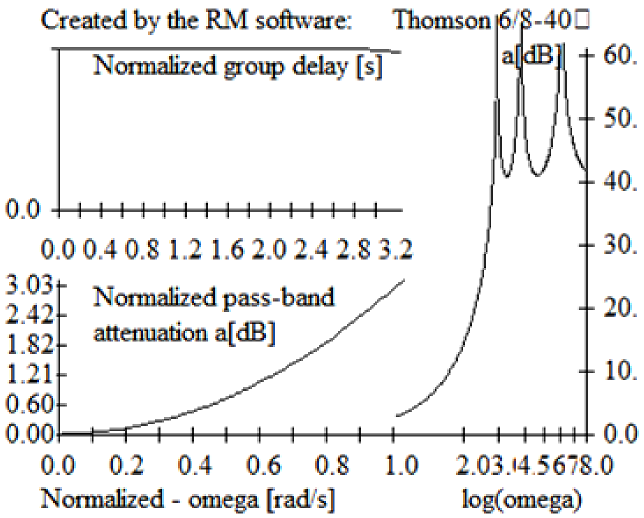


Fig 4. Attenuation and group delay response of the rational transfer function approximating constant group delay in maximally flat manner.

TABLE III. RATIONAL FUNCTION APPROXIMATING CONSTANT GROUP DELAY IN MAXIMALLY FLAT MANNER. N=6, M=8

| | Real part | Imaginary part |
|---|---|---|
| zeros | 0.000000 | ±2.939935 |
| | 0.000000 | ±3.834581 |
| | 0.000000 | ±6.096760 |
| poles | -2.902836 | ±0.450715 |
| | -2.703850 | ±1.359070 |
| | -1.474817 | ±3.300775 |
| | -2.269270 | ±2.293247 |

By inspection of Table I, one may come to a conclusion that by introduction of finite transmission zeros the transition region width was reduced for approximately 30% for the maximally flat case, and for approximately 18% for the equi-ripple case. That, however, brings both solutions very near to each other. The equi-ripple case has a very small advantage due to the larger error of the group delay approximation allowed.

## IV. TRANSFORMATION AND PROPERTIES OF THE NEW IIR FILTERS

The IIR filter was obtained by bilinear s-to-z transform of the analog prototypes. To preserve stability and reduce the effects of limited number of significant figures when representing the coefficients of the digital filter, the parallel structure of the IIR realization was adopted. It is depicted in Fig. 5.

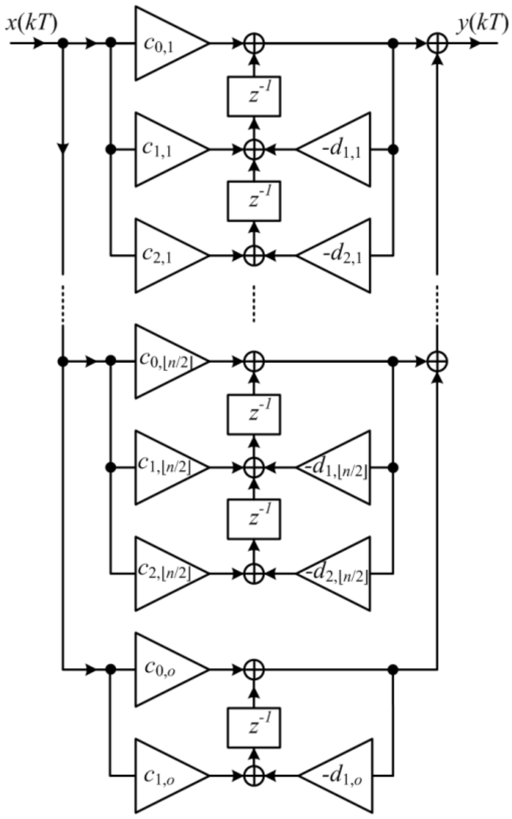


Fig. 5. Parallel realization of an odd-order IIR digital filter

Following is the procedure of obtaining the coefficients of Fig.5.

The original transfer function should be presented as a sum of partial fractions (of first (for $n$ odd) and second order terms) as follows

$$H(s)=\begin{cases} \sum_{i=0}^{m/2} H_e(s), & m-\text{even} \\ H_o(s)+\sum_{i=1}^{\lfloor m/2 \rfloor} H_e(s), & m-\text{odd} \end{cases} \tag{3}$$

where index $e$ is used for the complex pair of poles while $o$

means simple real pole. In (9) we used

$$H_e(s) = G_i \frac{s + b_{0,i}}{s^2 + a_{1,i}s + a_{0,i}} \tag{4}$$

with $G_i = 2\,\mathrm{re}\{r_i\}$, $b_{0,i} = -\left(\mathrm{re}\{p_i\} + \frac{\mathrm{im}\{r_i\}\mathrm{im}\{p_i\}}{\mathrm{re}\{r_i\}}\right)$,

$a_{1,i} = -2\,\mathrm{re}\{p_i\}$, $a_{i0} = |p_i|^2$, and

$$H_o(s) = G_o \frac{1}{s + a_o}, \tag{5}$$

with $G_o = r_o$, and $a_o = -p_o$. In the above $p_i$ stands for the $i$th pole, $r_i$ for the residue in the $i$th pole, "re" for "real part" and "im" for "imaginary part".

Implementing bilinear transform for the second order cell one gets (With reference to Fig. 5)

$$H_e(z) = \frac{c_{0,i} + c_{1,i}z^{-1} + c_{2,i}z^{-2}}{1 + d_{1,i}z^{-2} + d_{2,i}z^{-2}} \tag{6a}$$

with

$$c_{0,i} = G_i T \frac{b_{0,i}T + 2}{a_{0,i}T^2 + 2a_{1,i}T + 4}, \tag{6b}$$

$$c_{1,i} = G_i T \frac{2Tb_{0,i}}{a_{0,i}T^2 + 2a_{1,i}T + 4}, \tag{6c}$$

$$c_{2,i} = G_i T \frac{b_{0,i}T - 2}{a_{0,i}T^2 + 2a_{1,i}T + 4} \tag{6d}$$

$$d_{1,i} = \frac{2a_{0,i}T^2 - 8}{a_{0,i}T^2 + 2a_{1,i}T + 4}, \tag{6e}$$

and

$$d_{2,i} = \frac{a_{0,i}T^2 - 2a_{1,i}T + 4}{a_{0,i}T^2 + 2a_{1,i}T + 4}, \tag{6f}$$

while for the first order cell one has (With reference to Fig. 5.)

$$H_o(z) = \frac{c_{0,r} + c_{1,r}z^{-1}}{1 + d_{1,r}z^{-1}} \tag{7a}$$

with

$$c_{0,o} = c_{1,o} = G_o \frac{r_o T}{a_o T + 2}, \tag{7b}$$

and

$$d_{1,o} = \frac{a_o T - 2}{a_o T + 2}. \tag{7c}$$

In the next we will show the results of implementation of this procedure to the two example analog filters. Sampling frequency hundred times larger than the cut-off frequency of the filter was used in the bilinear transform.

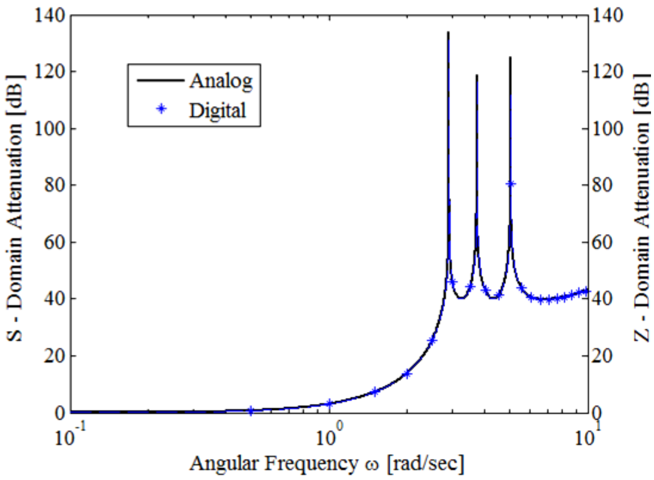


Fig. 6. Attenuation characteristics of the analog and the digital filter approximating in equi-ripple manner

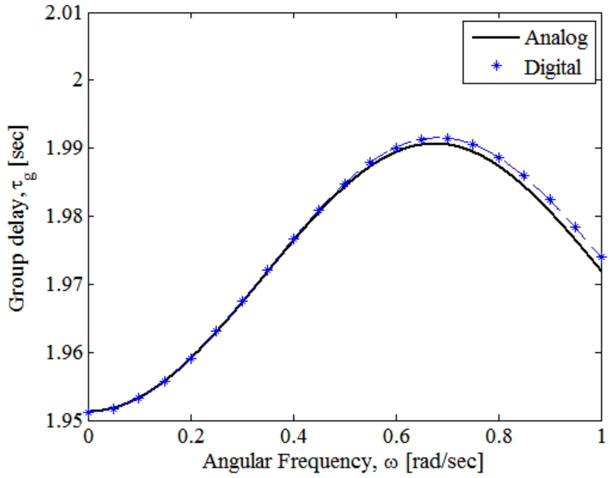


Fig. 7. Group delay characteristics of the analog and the digital filter approximating in equi-ripple manner

TABLE IV GROUP DELAY ERROR AS A CONSEQUENCE OF THE S-TO-Z TRANSFORMATION FOR THE FILTERS APROXIMATING IN EQUI-RIPPLE MANNER

| Group delay error @ ω = 1 | value | unit |
|---|---|---|
| $\Delta = \lvert\tau_{G(analog)} - \tau_{G(digital)}\rvert$ | 0.0019530953 | sec |
| $\Delta_r = 100 \cdot [\Delta / \tau_{G(analog)}]$ | 0.0990374792 | % |

TABLE V COEFFICIENTS OF Z – DOMAIN PARALLEL SECTIONS

| $i$ | $c_0$ | $c_1$ | $c_2$ |
|---|---|---|---|
| 1 | -0.05786132812500 | +0.00415039062500 | +0.06195068359375 |
| 2 | +0.06500244140625 | +0.00085449218750 | -0.06414794921875 |
| 3 | -0.00384521484375 | -0.00280761718750 | +0.00109863281250 |
| 4 | -0.00250244140625 | -0.00042724609375 | +0.00207519531250 |

| | $d_1$ | $d_2$ |
|---|---|---|
| 1 | -1.85571289062500 | +0.86248779296875 |
| 2 | -1.84570312500000 | +0.86566162109375 |
| 3 | -1.82989501953125 | +0.87463378906250 |
| 4 | -1.82427978515625 | +0.90191650390625 |

TABLE VI Z – DOMAIN ZEROS/POLE LOCATION OBTAINED AS THE ROOTS OF THE POLYNOMIALS GIVEN IN TABLE V

| Zeros | | Poles | |
|---|---|---|---|
| real part | imaginary part | real part | imaginary part |
| +1.0712206676 | 0.0 | +0.9278564453 | +0.0396258736 |
| -0.9994907098 | 0.0 | +0.9278564453 | -0.0396258736 |
| -1.0000000000 | 0.0 | +0.9228515625 | +0.1183495445 |
| +0.9868544601 | 0.0 | +0.9228515625 | -0.1183495445 |
| -1.0123793176 | 0.0 | +0.9149475098 | +0.1936616726 |
| +0.2822205874 | 0.0 | +0.9149475098 | -0.1936616726 |
| -1.0000000000 | 0.0 | +0.9121398926 | +0.2644188349 |
| +0.8292682927 | 0.0 | +0.9121398926 | -0.2644188349 |

The example related to the filter approximating group delay in equi-ripple manner will be discussed first.

Fig. 6 depicts the attenuation characteristics of both the original analog and the IIR digital filter obtained after transformation. As can be seen, no distortions introduced after the transformation may be seen.

The group delay characteristics of both analog and IIR digital filter are depicted in Fig. 7. Small distortion may be noticed. It is numerically highlighted in Table IV. One may find out from this table that the distortion of the group delay due to the bilinear transform is smaller than 1% over all pass-band.

For convenience, Table V contains the values of the coefficients of the parallel cells as defend with (6a) and Fig. 5. The corresponding zeros and poles of the cells are given in Table VI.

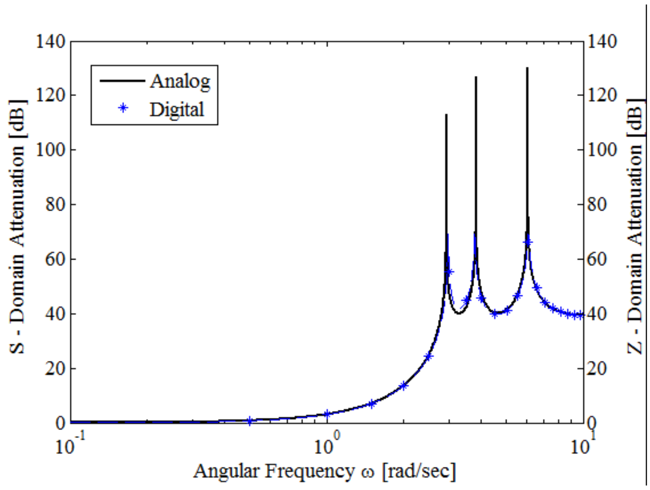


Fig. 8. Attenuation characteristics of the analog and the digital filter approximating in maximally flat manner

The second example is related to the filter approximating group delay in maximally flat manner.

Fig. 8 depicts the attenuation characteristics of both the original analog and the IIR digital filter obtained after transformation.

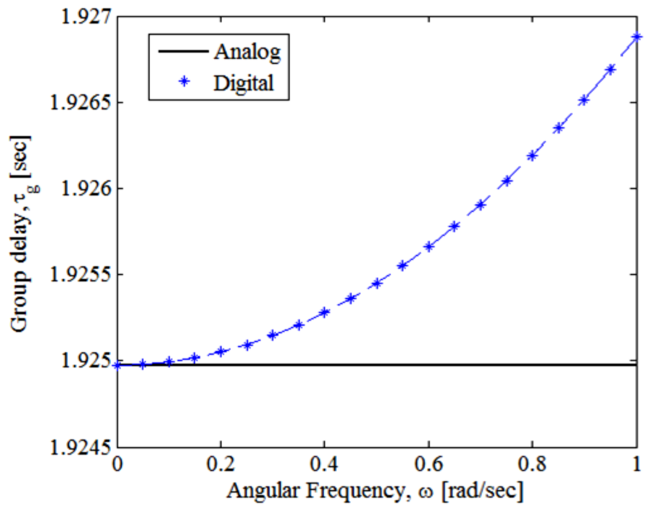


Fig. 9. Group delay characteristics of the analog and the digital filter approximating in maximally flat manner

TABLE VII GROUP DELAY ERROR AS A CONSEQUENCE OF THE S-TO-Z TRANSFORMATION FOR THE FILTERS APPROXIMATING IN MAXIMALLY FLAT MANNER

| Group delay error @ ω = 1 | value | unit |
|---|---|---|
| $\Delta = \lvert\tau_{G(analog)} - \tau_{G(digital)}\rvert$ | 0.0018991890 | sec |
| $\Delta_r = 100 \cdot [\Delta/\tau_{G(analog)}]$ | 0.0986604537 | % |

TABLE VIII COEFFICIENTS OF Z – DOMAIN PARALLEL SECTIONS FOR THE IIR FILTER APPROXIMATING IN MAXIMALLY FLAT MANNER

| $i$ | $c_0$ | $c_1$ | $c_2$ |
|---|---|---|---|
| 1 | -0.00933837890625 | +0.22723388671875 | +0.23663330078125 |
| 2 | +0.05548095703125 | -0.31170654296875 | -0.36718750000000 |
| 3 | -0.05621337890625 | +0.09002685546875 | +0.14624023437500 |
| 4 | +0.01068115234375 | -0.00402832031250 | -0.01464843750000 |
| | $d_1$ | $d_2$ | |
| 1 | -1.66510009765625 | +0.69366455078125 | |
| 2 | -1.68115234375000 | +0.71179199218750 | |
| 3 | -1.71697998046875 | +0.75262451171875 | |
| 4 | -1.78570556640625 | +0.83233642578125 | |

TABLE IX Z – DOMAIN ZEROS/POLE LOCATION OBTAINED AS THE ROOTS OF THE POLYNOMIALS GIVEN IN TABLE VIII

| Zeros | | Poles | |
|---|---|---|---|
| real part | imaginary part | real part | imaginary part |
| +25.3335815315 | 0.0 | +0.8325500488 | +0.0229121579 |
| -1.0002481982 | 0.0 | +0.8325500488 | -0.0229121579 |
| +6.6182618262 | 0.0 | +0.8405761719 | +0.0722751096 |
| -1.0000000000 | 0.0 | +0.8405761719 | -0.0722751096 |
| +2.6015200869 | 0.0 | +0.8584899902 | +0.1249777916 |
| -1.0000000000 | 0.0 | +0.8584899902 | -0.1249777916 |
| +1.3747365752 | 0.0 | +0.8928527832 | +0.1874842215 |
| -0.9975937181 | 0.0 | +0.8928527832 | -0.1874842215 |

As can be seen, no distortions introduced after the transformation may be seen.

The group delay characteristics of both analog and IIR digital filter are depicted in Fig. 9. Small distortion may be noticed. It is numerically highlighted in Table VII. One may find out from this table that the distortion of the group delay due to the bilinear transform is smaller than 1% over all pass-band.

For convenience, Table VIII contains the values of the coefficients of the parallel cells as defend with (6a) and Fig. 5. The corresponding zeros and poles of the cells are given in Table IX.

Assuming direct hardware implementation i.e. each element of the architecture illustrated in Fig. 5 realized as separate, two port component, one can easily determine the amount of hardware required for filter realization. Table X summarizes hardware estimation for both, equi-ripple and maximally flat, cases.

TABLE X AMOUNT OF HARDWARE REQUIRED FOR FILTERS EQUI-RIPPLE AND MAXIMALLY FLAT FILTERS

| Component | number |
|---|---|
| Adders | 19 |
| Multipliers | 20 |
| Registers (Delay Elements) | 8 |

## V. CONCLUSION

An overview of the procedure of design of selective linear-phase IIR digital filters was described and exemplified. Synthesis of minimum-phase transfer functions was considered. Two approximation criteria have being implemented for constant group delay approximation: the maximally flat and the equi-ripple. The synthesis in the s-domain was performed in two steps. First, a polynomial function exhibiting linear phase was synthesized and then transmission zeros located on the imaginary axis were added. The IIR digital filter was obtained by bilinear transform. To avoid stability problems and, in the same time, problems related to the signal dynamics within the filter, parallel implementation of the IIR filter was adopted. Two examples were be given exemplifying the implementation of the two approximation criteria used.

## ACKNOWLEDGMENT

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## REFERENCES

[1] Williams A. and Taylor F., *Electronic Filter Design Handbook*, McGraw-Hill Professional, 4 edition, 2006.

[2] Thajchayapong, P. and Khunkitti, A., "Recursive digital filters with predetermined group delay and Chebyshev stopband attenuation", Electronics Letters ,Vol. 24 , Issue: 25 , 8 Dec 1988, pp. 1547 - 1549.

[3] Mollova, G.S. and Unbehauen R., "Design of recursive filters with constant group delay and Chebyshev attenuation", Proc. of the 1998 IEEE Int. Conf. on Acoustics, Speech and Signal Processing, 1998, May 1998, Seattle, WA , pp. 1257 - 1260 vol.3.

[4] Okoniewski, P. and Piskorowski, J., "An analytical approach to the group delay compensation of digital IIR filters," Methods and Models in Automation and Robotics (MMAR), 2012 17th International Conference on, vol., no., pp.75,78, 27-30 Aug. 2012. doi: 10.1109/MMAR.2012.6347908

[5] Piskorowski, J., Kaszynski, R., de Anda, M. and Sarmiento-Reyes, A., "Group delay compensation and settling time minimization in continuous-time elliptic filters," Electrotechnical Conference, 2008. MELECON 2008. The 14th IEEE Mediterranean , vol., no., pp.12,16, 5-7 May 2008. doi: 10.1109/MELCON.2008.4618403.

[6] Deczky A.G., "Recursive Digital Filters Having Equiripple Group Delay", *IEEE Trans. Circuits Syst.,* Vol. CAS-2 1, 1974, pp. 13 l-134.

[7] Fettweis A., "A Simple Design of Maximally Flat Delay Digital Filters", *IEEE Trans. Audio and Electroacoust.,* Vol. 20, pp.1 12-l 14, 1972.

[8] Thiran J.P., "Recursive Digital Filters with Maximally Flat Group Delay", *IEEE Trans. Circuit Theory,* vol. 18, pp. 659-664, 1971.

[9] W.E., Thomson, "Delay Networks having Maximally Flat Frequency Characteristics", *Proceedings of the Institution of Electrical Engineers*, Part III, November 1949, Vol. 96, No. 44, pp. 487–490.

[10] Litovski V., "Gaussian filter with a modified characteristic function", Proc. of the XX Konf. ETAN-a, Opatija, 1976, (in Serbian).